Hilbert 16, Riemann Mapping Theorem, Dirichlet problem and o-minimality

I. Hilbert 16

Hilbert’s 16th problem, part 2

Does a polynomial vector field on the real plane have only finitely many limit cycles?

Is the number of limit cycles of a polynomial vector field on the real plane bounded by a constant depending on the degree of the polynomials only?

How does such an upper bound explicitly look like?
Basic definitions

1) Vector field

\[ U \subset \mathbb{R}^n \text{ open, } v : U \to \mathbb{R}^n \text{ } C^1\text{-map} \]

Differential equation (ODE) given by the vector field \( v \):

\[ \dot{x} = v(x) \]

Trajectory:

A trajectory (of \( v \)) is a \( C^1 \)-map \( \gamma : I \to U \) (with \( I \) a maximal interval) which solves the ODE \( \dot{x} = v(x) \), i.e.

\[ \frac{d\gamma}{dt}(t) = v(\gamma(t)) \quad \text{for all} \quad t \in I. \]
Orbit:

An orbit is the image of a trajectory. Two orbits are equal or disjoint, each point is in an orbit.

→ Phase portrait

2) Limit cycle (plane case, $U = \mathbb{R}^2$)

Closed trajectory:

A trajectory $\gamma : I \rightarrow \mathbb{R}^2$ is closed if there are $t_1 < t_2$ with $\gamma(t_1) = \gamma(t_2)$ (and then $I = \mathbb{R}$).
Closed orbit:

A closed orbit is the orbit of a closed trajectory

\[
\text{Closed orbit} = \text{Compact orbit}
\]

Cycle:

A cycle is a closed orbit which is not a singleton.

\(\{x\} \text{ is an orbit of } v \iff v(x) = 0, \text{ i.e. } x \text{ is a singular point of the vector field}\)

Limit cycle:

A cycle \(\Gamma\) is called a limit cycle if there is no neighbourhood of \(\Gamma\) which contains only cycles.
Examples

\[ \dot{x} = -y \]
\[ \dot{y} = x \]

No limit cycles

Limit cycle

“Limit cycles determine the topology of the phase portrait”
Hilbert 16, part 2

\[ \dot{x} = p(x, y), \quad \dot{y} = q(x, y), \quad p, q \in \mathbb{R}[x, y] \]

History of Hilbert 16, part 2

Question 2 and 3 are still open.

Question 1 was answered positively by H. Dulac in 1923.

But the proof contained a gap!

The problem was solved independently by Y. Ilyashenko and J. Ecalle in 1992.
Idea of Dulac /Ilyashenko

Assume that there are infinitely many limit cycles. Then they converge to a polycycle.

Polycycle

A polycycle is a finite collection of singular points and orbits between them.

Choose a cross section and consider the Poincaré map, i.e. the first return map. It is the
composition of finitely many transition maps and analytic maps.

Closed orbit = Fixed point of the Poincaré map

Limit cycle = Not interior point of the set of fixed points

“Poincaré map qualitatively codes the phase portrait of the ODE”

Solution to Question 1

Poincaré map has infinitely many fixed points
⇒ Poincaré map is the identity
II. Riemann Mapping Theorem

Riemann Mapping Theorem

Every simply connected domain $\Omega \subsetneq \mathbb{C}$ can be mapped biholomorphically onto the unit ball $B(0, 1) := \{z \in \mathbb{C} \mid |z| < 1\}$.

Uniqueness:

Let $a \in \Omega$. There is exactly one biholomorphic map $\varphi : \Omega \rightarrow B(0, 1)$ with $\varphi(a) = 0$ and $\varphi'(a) > 0$.

Examples

1) $\text{Aut } B(0, 1) = \{z \mapsto \rho \frac{z-a}{\overline{a}z-1} \mid |a| < 1, |ho| = 1\}$.

2) $\mathbb{H} := \{z \in \mathbb{C} \mid \text{Im } z > 0\}$ upper half-plane: $\mathbb{H} \overset{\mathbb{R}}{\rightarrow} B(0, 1)$, $z \mapsto \frac{z-i}{z+i}$.

"Möbius transformations"
III. Dirichlet problem

We consider the Laplace equation with boundary value problem, called Dirichlet problem.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $g : \partial \Omega \to \mathbb{R}$ be continuous. We look for a function $u : \Omega \to \mathbb{R}$ with $u \in C^2(\Omega) \cap C^0(\Omega)$ such that

$$\Delta u = 0 \quad \text{in} \quad \Omega,$$
$$u = g \quad \text{on} \quad \partial \Omega.$$

(where $\Delta := \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ Laplace operator).

Very important PDE with many applications in physics:

Physical interpretation:

$g$ given temperature on the boundary $\Rightarrow u$ temperature in $\Omega$
IV. Asymptotic expansion

1. For the Poincaré map

Definition

A polycycle is called elementary if all its singular points are elementary. A singular point (of a vector field) is called elementary if its linear part (in the Taylor expansion) has at least one nonzero eigenvalue.

Theorem (Dulac)

The Poincaré map $P(x)$ of an elementary polycycle of a polynomial vector field (resp. the transition map $T(x)$ at an elementary singular point) is either flat, inverse to flat or has an asymptotic expansion

$$P(x) \sim \sum_{n=0}^{\infty} a_n P_n(\log x)x^{\alpha_n},$$
with \((\alpha_n) \subset \mathbb{R}_{>0}, \alpha_n \uparrow \infty, (a_n) \subset \mathbb{R}, (P_n) \subset \mathbb{R}[x]\), and for each \(N\)

\[
P(x) - \sum_{n=0}^{N} a_n P_n(\log x)x^{\alpha_n} = o(x^{\alpha_N}),
\]

\[
P(x) - \sum_{n=0}^{N} a_n P_n(\log x)x^{\alpha_n}
\]

i.e. \(\lim_{x \to 0} \frac{P(x) - \sum_{n=0}^{N} a_n P_n(\log x)x^{\alpha_n}}{x^{\alpha_N}} = 0\).

(Here the cross section is identified with the real line, the intersection point of the cross section and the polycycle with 0.)

- \(P(x)\) flat \(\Rightarrow\) \(P(x)\) has no fixed points near 0

- \(P(x)\) inverse to flat \(\Rightarrow\) -""-

- Case of asymptotic expansion: We want to argue as follows:
\( P(x) \) has infinitely many fixed points
\[ \Rightarrow P(x) \sim x \]
\[ ! \Rightarrow P(x) = x \]
\[ ! : \text{Quasianalyticity} \]

Construction of quasianalyticity:

**Riemann surface of the logarithm**

\( \mathcal{L} = \mathbb{R}_{>0} \times \mathbb{R} \). \((r, \varphi) \in \mathcal{L} : r \text{ radius}, \varphi \text{ angle} \)

\( \mathcal{L} \) is Riemann surface via the projection map

\[ \log : \mathcal{L} \to \mathbb{C}, (r, \varphi) \mapsto \log r + i\varphi . \]

**Definition**

- \( U \subset \mathcal{L} \) is called a standard quadratic domain if there are \( c, C > 0 \) such that

\[ U = \{(r, \varphi) \in \mathcal{L} \mid 0 < r < c \exp(-C \sqrt{|\varphi|}) \} . \]
\[ Q := \{ f : U \to \mathbb{C} \text{hol.} \mid U \subset \mathcal{L} \text{ standard quadratic domain and } f \text{ has an asymptotic expansion at } 0 \}. \]

Asymptotic development at 0:

\[ \exists (\alpha_n) \in \mathbb{R}_{>0}, \alpha_n \nearrow \infty, (a_n) \subset \mathbb{C}, (P_n) \subset \mathbb{C}[z] \]

\[ \text{s.t. } \forall N \in \mathbb{N}_0 \]

\[ f(z) - \sum_{n=0}^{N} a_n P_n(\log z) z^{\alpha_n} = o(z^{\alpha_N}). \]

(Here \( z^\alpha : \mathcal{L} \to \mathbb{C}, z \mapsto e^{\alpha \log z} \).)

**Theorem** (Ilyashenko)

a) \( Q \) is a quasianalytic ring, i.e. if \( f \in Q \) has asymptotic expansion 0, then \( f = 0 \).

b) \( T(x) \in Q \) (if \( T(x) \) not flat or inverse to flat)
Remark

The generic case of a transition map $T(x)$ at a singular point is given when the singular point is a hyperbolic saddle (the linear part has two real eigenvalues, one $> 0$, one $< 0$).

If the ratios of the two eigenvalues of a singular point is irrational, then no log-terms occur:

$$T(x) \sim \sum_{n=0}^{\infty} a_n x^{\alpha_n}.$$  

2. For the Riemann map and the Dirichlet solution.

$\Omega \subset \mathbb{R}^2$ denotes a bounded semianalytic domain in the plane.
Theorem (K.)

Assume that $\Omega$ is simply connected. Let $x \in \partial \Omega$ with $\langle x, \partial \Omega \rangle > 0$. Then the germ of the Riemann map (i.e. biholomorphic map) $\Omega \xrightarrow{\approx} B(0, 1)$ at $x$ is an element of $Q$.

If $\langle x, \partial \Omega \rangle \in \pi(\mathbb{R} \setminus \mathbb{Q})$, no log-terms occur, if $\langle x, \partial \Omega \rangle \in \pi \mathbb{Q}$, log-terms occur.

Theorem (K.)

Let $g \in \partial \Omega \to \mathbb{R}$ be a continuous function on the boundary, which is semianalytic. Let $x \in \partial \Omega$ with $\langle x, \partial \Omega \rangle > 0$. Then the germ of the Dirichlet solution $u$ for $g$, i.e.

$$\Delta u = 0 \quad \text{in} \quad \Omega,$$
$$u = g \quad \text{on} \quad \partial \Omega,$$

at $x$ is the real part of an element of $Q$.

If $\langle x, \partial \Omega \rangle \in \pi(\mathbb{R} \setminus \mathbb{Q})$, no log-terms occur, if $\langle x, \partial \Omega \rangle \in \pi \mathbb{Q}$, log-terms occur.
V. Poincaré map, Riemann map, Dirichlet solution

In joint work with J.-P. Rolin and P. Speissegger we could show o-minimality under the assumption that the asymptotic expansion has no log-terms:

Definition

\[ Q_0 := \{ f : U \to \mathbb{C} \text{ hol.} \mid U \subset \mathcal{L} \text{ standard quadratic domain and } f \text{ has an asymptotic expansion at } 0 \text{ without log-terms} \} \]

Theorem (K., Rolin, Speissegger)

The germs of functions in \( Q_0 \) at 0, restricted to the positive real line (\( \mathbb{R}_{>0} = \mathbb{R}_{>0} \times \{0\} \subset \mathbb{R}_{>0} \times \mathbb{R} = \mathcal{L} \)) are definable in an o-minimal structure, denoted by \( \mathbb{R}_Q \).
**Theorem** (K., Rolin, Speissegger)

The transition map at a hyperbolic singularity with irrational ratio of the eigenvalues of a polynomial vector field is definable in the o-minimal structure $\mathbb{R}_Q$.

**Theorem** (K.)

Let $\Omega \subset \mathbb{C}$ be a simply connected domain which is bounded and semianalytic.

Assume that the angles of $\partial \Omega$ at (the finitely many) singular boundary points have size $\pi \cdot$ (irrational number).

Then the Riemann map $\Omega \xrightarrow{\sim} B(0,1)$ is definable in the o-minimal structure $\mathbb{R}_Q$.

**Theorem** (K.)
Let $\Omega \subset \mathbb{R}^n$ be a bounded and semianalytic domain.

Assume that the angles of $\partial \Omega$ at (the finitely many) singular boundary points have size $\pi \cdot (\text{irrational number})$.

Let $g : \partial \Omega \to \mathbb{R}$ be continuous and semianalytic.

Then the Dirichlet solution of $g$ is definable in the o-minimal structure $\mathbb{R}_{Q, \exp}$. 